MATHEMATICAL ADDENDA TO FALLING: A BEGINNER'S GUIDE

NOT_CUB

1. The problem (by Jonny Nexus)

Here it is...

Rules for falling:

There are two "tests" you have to pass to avoid unconsciousness.

I want to know the overall percentage change of a character falling unconscious.

FIRST TEST

Each character has a number of hit points (rated at 1 upwards).

The character first takes 20d6 points of damage (i.e. 20 six sided dice rolled and then added together). So you will roll a minimum of 20 points and a maximum of 120 points.

If you go below zero then you are knocked out.

i.e. a character has 75 points. If he rolls 74 he is still conscious. If he rolls 76 then he will fall unconscious.

SECOND TEST

If the character is still conscious, after applying hit point damage, AND they took more than 50 points of damage, they then they have to save verses massive damage (they die if they fail).

(If they took 50 points or less, then they're okay, passed).

Each character has a Fortitude Save rating. This is a modifier rated from 0 upward to plus whatever. To do a Fortitude Save you roll a twenty sided dice and add your Fortitude Save to it. You have to get a number of 15 or greater.

Example:

A character has a Fortitude Save of +6. This means that he must roll a 9 or greater. If he rolls 8 or less then he is down and out.

SUMMARY EXAMPLES

A character has 75 hitpoints and +6 Fortitude Save.

EXAMPLE 1: He rolls 20 6-sided dice which total 76. He has failed.

EXAMPLE 2: He rolls 20 6-sided dice which total 50. He has succeeded.

EXAMPLE 3: He rolls 20 6-sided dice which total 74. He then rolls 8 on a 20-sided dice making a total of 14. He has failed.

EXAMPLE 4: He rolls 20 6-sided dice which total 74. He then rolls 9 on a 20-sided dice making a total of 15. He has succeeded.

I need the a formula which will tell me his chances of succeeding in percentage terms.

I'm not sure how coherant the above is, so have a scan through it and see what you think. I imagine it's quite tricky because it's two probabilities and a condition.

2. Solution for the case of a character with 75 hitpoints

Definition 1. *Definition of $\mathbb{P}$, the probability symbol.*

By $\mathbb{P}(X_i = k)$ I mean the probablity that the ith roll is equal to k expressed as a fraction (between 0 and 1, not a percentage). So X_i takes the value of the ith roll.

Example 2. *Example of the use of $\mathbb{P}$*

So $\mathbb{P}(X_2 = 3) = \frac{1}{6}$ since the probability that the second roll is equal to 3 is 1 in 6. X_i is known as a random variable.

Definition 3. *Definition of a probability generating function*

The probability generating function $G_X(s)$ is defined by $G_X(s) = \sum_k \mathbb{P}(X = k)s^k$ for any given random variable X, where $\sum_k$ denotes the sum over all possible values of k.

Example 4. *Example of a probablity generating function*

So for example, if Y is a random variable that takes value 1 with probability $\frac{2}{3}$ and value 2 with probablity $\frac{1}{3}$, then $G_Y(s) = \frac{2}{3}s^1 + \frac{1}{3}s^2$

Example 5. *The probability generating function of a random variable that represents rolling a dice.*

Using definitions 1 and 3, we have $G_{X_i}(s) = \frac{1}{6}(s + s^2 + s^3 + s^4 + s^5 + s^6)$.

Theorem 6. *The probability generating function of a sum of random variables (which is itself a random variable)*

If Z is a random variable defined by $Z = X_1 + \cdots + X_n$, then

$$G_Z(s) = \prod_{i=1}^{n} G_{X_i}(s)$$

where $\prod$ denotes the product as i runs from 1 to n. Equivalent to (pseudocode because I don't want to look up function pointers right now):

```
function prod(function, start, end)
begin
  result:=1
  for i:=start to end do
    result:=result*function(i);
end;
```

[No proofs will be given]

This is true for any set of random variables X_i, but of particular use in our case, since Z is the sum of the 20 rolls (if we take $n = 20$ in the theorem). This is useful, because we can then expand out this product, and look at coefficients of s^i for each i, and get probabilities back from this using definition 3.

Example 7. *Reading probabilities back out of a probability generating function*

If we have $G_Y(s) = \frac{1}{2}s^1 + \frac{1}{2}s^2$, then we can get $\mathbb{P}(Y = 1) = \frac{1}{2}, \mathbb{P}(Y = 2) = \frac{1}{2}$.

Example 8. *The probablity of getting 69 when as the sum of 20 6-sided dice*

Using example 5 and theorem 6 we have $G_Z = (\frac{1}{6}(s + s^2 + s^3 + s^4 + s^5 + s^6))^{20}$
It's going to be somewhat boring to expand this by hand, so I have had the computer do it for us in section 4

So from $\cdots + 3914382207515/76169967501312 s^{69} + \cdots$ and example 7, we see that, for example $\mathbb{P}(Z = 69) = \frac{3914382207515}{76169967501312}$. That is the probability that the total of the dice thrown is 69.

Definition 9. *Tail generating functions*

The tail generating function is defined as

$$T_X(s) = \sum_n \mathbb{P}(X > n)s^n$$

where $\sum_n$ denotes the sum over all possible values of n.

Theorem 10. *Working out a tail generating function given a probability generating function*

$$(1 - s)T_X(s) = 1 - G_X(s) \quad \text{if } X \geq 0$$

.

Example 11. *Working out the probability of throwing more than 75 and more than 50 with 20 6-sided dice*

So using theorem tgfdefn and example 7

$$T_Z(s) = \frac{1 - (\frac{1}{6}(s + s^2 + s^3 + s^4 + s^5 + s^6))^{20}}{1 - s}$$

Again, the computer can expand this for us in section 4

So $\mathbb{P}(Z > 75) = \frac{144442487964125}{609359740010496} \simeq 0.237 = 23.7\%$ since this is the coefficient of s^{75} in the above mess, using definition 8. $\mathbb{P}(Z > 50) = \frac{1819063241546635}{1828079220031488} \simeq 0.995 = 99.5\%$ similarly. So from test 1, you have a 0.5% chance of living "instantly". A 23.7% chance of dying instantly, and a 75.8% chance of needing to make the save roll.

Example 12. *Probability of making the fortitude roll*

Now, let Y be the random variable representing a roll on a D20. Then $\mathbb{P}(Y \geq n) = \frac{21-n}{20}$ for $0 \leq n \leq 20$, intuitively (try out a few values of n and you'll see what I mean). He has to roll $15 - F$ or greater, where F is his fortitude save. So $\mathbb{P}(Y \geq 15 - F) = \frac{21-(15-F)}{20} = \frac{F+6}{20}$, for $F < 15$, and 1 if $F \geq 15$.

His chance of not making his fortitude roll is $1 - \frac{F+6}{20} = \frac{14-F}{20}$ for $F < 15$ and 0 if $F \geq 15$.

Example 13. *Putting it all together*

Using the results of examples 11 and 12: So, if $F \geq 15$, then he will definitely make his save, and he has a $(0.5 + 75.8)\% = 76.3\%$ chance of living.

If $0 \leq F < 15$ then, he will make his save with probability $\frac{F+6}{20}$, so he has a $(0.5 + 75.8\frac{F+6}{20})\%$ chance of living. eg if his fortitude is 6, he has a $(0.5 + 75.8\frac{6+6}{20})\% \simeq 46.0\%$ chance of living.

n	f(n)	n	f(n)	n	f(n)	n	f(n)
20	1.0	45	0.999	70	0.474	95	3.09e-4
21	1.0	46	0.999	71	0.423	96	1.8e-4
22	1.0	47	0.999	72	0.373	97	1.02e-4
23	1.0	48	0.998	73	0.325	98	5.64e-5
24	1.0	49	0.997	74	0.279	99	3.02e-5
25	1.0	50	0.995	75	0.237	100	1.58e-5
26	1.0	51	0.993	76	0.199	101	7.96e-6
27	1.0	52	0.99	77	0.164	102	3.89e-6
28	1.0	53	0.985	78	0.134	103	1.84e-6
29	1.0	54	0.979	79	0.108	104	8.34e-7
30	1.0	55	0.972	80	0.0851	105	3.64e-7
31	1.0	56	0.962	81	0.0663	106	1.52e-7
32	1.0	57	0.949	82	0.0509	107	6.05e-8
33	1.0	58	0.934	83	0.0384	108	2.29e-8
34	1.0	59	0.915	84	0.0285	109	8.16e-9
35	1.0	60	0.893	85	0.0208	110	2.73e-9
36	1.0	61	0.866	86	0.0149	111	8.49e-10
37	1.0	62	0.836	87	0.0105	112	2.43e-10
38	1.0	63	0.802	88	0.00727	113	6.3e-11
39	1.0	64	0.763	89	0.00493	114	1.45e-11
40	1.0	65	0.721	90	0.00328	115	2.91e-12
41	1.0	66	0.676	91	0.00214	116	4.85e-13
42	1.0	67	0.628	92	0.00136	117	6.32e-14
43	1.0	68	0.577	93	8.51e-4	118	5.75e-15
44	0.999	69	0.526	94	5.2e-4	119	2.74e-16

Table 1. Values of $f(n)$

3. Solution for the case of a character with an arbitrary number of hitpoints

The solution given in examples 11 and 13 is easily modified to a character with H hitpoints. Define $f(n)$ to mean the coefficient of s^n in $T_Z(s)$; that is, the probability of rolling more than n with 20 6-sided dice. Then reworking example 11, if $H \leq 50$, he will either live or die, without needing to make a fortitude roll. Also, if $F \geq 15$, he will certainly make his fortitude roll, so he might as well not bother. In either case, the chance of him dying is equal to the chance of rolling more than H, so

$$\mathbb{P}(\text{dying}) = f(H)$$

and

$$\mathbb{P}(\text{living}) = 1 - f(H)$$

If $H > 50$, then he will die either if he rolls more than H, or if he rolls between 50 and H, and fails to make the fortitude roll. So, if $F < 15$:

$$\mathbb{P}(\text{dying}) = f(H) + (f(50) - f(H))\frac{14 - F}{20}$$

and

$$\mathbb{P}(\text{living}) = 1 - \left(f(H) + (f(50) - f(H))\frac{14 - F}{20}\right)$$

4. Computer polynomial calculations

This is a session from mupad. The computer works out the coefficients of the large polynomial for us.

```
>> DIGITS:=3:
>> GXi:=s->(s+s^2+s^3+s^4+s^5+s^6)/6:
>> GX:=s->GXi(s)^20:
>> T:=s->(1-GX(s))/(1-s):
>> f:=n->coeff(simplify(T(s)), s, n):
>> expand(GX(s))
1/3656158440062976*s^20 + 5/914039610015744*s^21 + 35/609359740010496*s^22\
 + 385/914039610015744*s^23 + 8855/3656158440062976*s^24 + 1771/1523399350\
02624*s^25 + 22135/457019805007872*s^26 + 82175/457019805007872*s^27 + 738\
625/1218719480020992*s^28 + 1719025/914039610015744*s^29 + 9926455/1828079\
220031488*s^30 + 4481435/304679870005248*s^31 + 137578715/3656158440062976\
*s^32 + 41777675/457019805007872*s^33 + 16134325/76169967501312*s^34 + 214\
765265/457019805007872*s^35 + 14300825/14281868906496*s^36 + 313042765/152\
339935002624*s^37 + 3720967315/914039610015744*s^38 + 3565343875/457019805\
007872*s^39 + 1471351535/101559956668416*s^40 + 3978067225/152339935002624\
*s^41 + 1742288525/38084983750656*s^42 + 3960501545/50779978334208*s^43 + \
157876934815/1218719480020992*s^44 + 63931449413/304679870005248*s^45 + 67\
385598265/203119913336832*s^46 + 468341482015/914039610015744*s^47 + 28285\
41558365/3656158440062976*s^48 + 174074868515/152339935002624*s^49 + 75500\
2119635/457019805007872*s^50 + 1069011841375/457019805007872*s^51 + 197749\
3247335/609359740010496*s^52 + 1008546361265/228509902503936*s^53 + 672589\
341665/114254951251968*s^54 + 586732060573/76169967501312*s^55 + 180832595\
79515/1828079220031488*s^56 + 5699289962305/457019805007872*s^57 + 2351813\
952475/152339935002624*s^58 + 8578992730375/457019805007872*s^59 + 8198700\
9993775/3656158440062976*s^60 + 8020045695295/304679870005248*s^61 + 55515\
355578395/1828079220031488*s^62 + 31478097575165/914039610015744*s^63 + 46\
792067208125/1218719480020992*s^64 + 19234413266555/457019805007872*s^65 +\
 2591536192775/57127475625984*s^66 + 2441745318785/50779978334208*s^67 + 3\
0544362152285/609359740010496*s^68 + 3914382207515/76169967501312*s^69 + 2\
631346887493/50779978334208*s^70 + 3914382207515/76169967501312*s^71 + 305\
44362152285/609359740010496*s^72 + 2441745318785/50779978334208*s^73 + 259\
1536192775/57127475625984*s^74 + 19234413266555/457019805007872*s^75 + 467\
92067208125/1218719480020992*s^76 + 31478097575165/914039610015744*s^77 + \
55515355578395/1828079220031488*s^78 + 8020045695295/304679870005248*s^79 \
+ 81987009993775/3656158440062976*s^80 + 8578992730375/457019805007872*s^8\
1 + 2351813952475/152339935002624*s^82 + 5699289962305/457019805007872*s^8\
3 + 18083259579515/1828079220031488*s^84 + 586732060573/76169967501312*s^8\
5 + 672589341665/114254951251968*s^86 + 1008546361265/228509902503936*s^87\
 + 1977493247335/609359740010496*s^88 + 1069011841375/457019805007872*s^89\
 + 755002119635/457019805007872*s^90 + 174074868515/152339935002624*s^91 +\
 2828541558365/3656158440062976*s^92 + 468341482015/914039610015744*s^93 +\
 67385598265/203119913336832*s^94 + 63931449413/304679870005248*s^95 + 157\
876934815/1218719480020992*s^96 + 3960501545/50779978334208*s^97 + 1742288\
525/38084983750656*s^98 + 3978067225/152339935002624*s^99 + 1471351535/101\
559956668416*s^100 + 3565343875/457019805007872*s^101 + 3720967315/9140396\
```

```
10015744*s^102 + 313042765/152339935002624*s^103 + 14300825/14281868906496\
*s^104 + 214765265/457019805007872*s^105 + 16134325/76169967501312*s^106 +\
 41777675/457019805007872*s^107 + 137578715/3656158440062976*s^108 + 44814\
35/304679870005248*s^109 + 9926455/1828079220031488*s^110 + 1719025/914039\
610015744*s^111 + 738625/1218719480020992*s^112 + 82175/457019805007872*s^\
113 + 22135/457019805007872*s^114 + 1771/152339935002624*s^115 + 8855/3656\
158440062976*s^116 + 385/914039610015744*s^117 + 35/609359740010496*s^118 \
+ 5/914039610015744*s^119 + 1/3656158440062976*s^120
>> simplify(T(s))
s + s^2 + s^3 + s^4 + s^5 + s^6 + s^7 + s^8 + s^9 + s^10 + s^11 + s^12 + s\
^13 + s^14 + s^15 + s^16 + s^17 + s^18 + s^19 + 3656158440062975/365615844\
0062976*s^20 + 1218719480020985/1218719480020992*s^21 + 1218719480020915/1\
218719480020992*s^22 + 3656158440061205/3656158440062976*s^23 + 6093597400\
08725/609359740010496*s^24 + 609359740001641/609359740010496*s^25 + 182807\
9219916383/1828079220031488*s^26 + 609359739862561/609359740010496*s^27 + \
1218719478986497/1218719480020992*s^28 + 3656158430083391/3656158440062976\
*s^29 + 1218719470076827/1218719480020992*s^30 + 1218719452151087/12187194\
80020992*s^31 + 1828079109437273/1828079220031488*s^32 + 609359647442191/6\
09359740010496*s^33 + 609359518367591/609359740010496*s^34 + 1828077696041\
713/1828079220031488*s^35 + 609358621845371/609359740010496*s^36 + 6093573\
69674311/609359740010496*s^37 + 1828064667088303/1828079220031488*s^38 + 2\
03116711745867/203119913336832*s^39 + 203113769042797/203119913336832*s^40\
 + 609325394859491/609359740010496*s^41 + 203099172747697/203119913336832*\
s^42 + 203083330741517/203119913336832*s^43 + 1218342107514287/12187194800\
20992*s^44 + 406028793905545/406239826673664*s^45 + 405894022709015/406239\
826673664*s^46 + 3651172838453075/3656158440062976*s^47 + 608057382815785/\
609359740010496*s^48 + 607361083341725/609359740010496*s^49 + 181906324154\
6635/1828079220031488*s^50 + 604929064727045/609359740010496*s^51 + 301475\
785739855/304679870005248*s^52 + 900393171774505/914039610015744*s^53 + 29\
8337485680395/304679870005248*s^54 + 295990557438103/304679870005248*s^55 \
+ 1757860085049103/1828079220031488*s^56 + 578354308399961/609359740010496\
*s^57 + 568947052590061/609359740010496*s^58 + 1672525186848683/1828079220\
031488*s^59 + 1087687787901197/1218719480020992*s^60 + 1055607605120017/12\
18719480020992*s^61 + 3055792104203261/3656158440062976*s^62 + 97662657130\
0867/1218719480020992*s^63 + 464917252046371/609359740010496*s^64 + 131781\
4103072893/1828079220031488*s^65 + 137209438322677/203119913336832*s^66 + \
127442457047537/203119913336832*s^67 + 175891504495163/304679870005248*s^6\
8 + 53411325221701/101559956668416*s^69 + 48148631446715/101559956668416*s\
^70 + 128788365510085/304679870005248*s^71 + 75677456289295/20311991333683\
2*s^72 + 65910475014155/203119913336832*s^73 + 510265116958595/18280792200\
31488*s^74 + 144442487964125/609359740010496*s^75 + 242092908720125/121871\
9480020992*s^76 + 600366335859715/3656158440062976*s^77 + 163111874900975/\
1218719480020992*s^78 + 131031692119795/1218719480020992*s^79 + 1555540331\
82805/1828079220031488*s^80 + 40412687420435/609359740010496*s^81 + 310054\
31610535/609359740010496*s^82 + 70219134982385/1828079220031488*s^83 + 868\
9312567145/304679870005248*s^84 + 6342384324853/304679870005248*s^85 + 136\
46438241239/914039610015744*s^86 + 3204084265393/304679870005248*s^87 + 44\
30675283451/609359740010496*s^88 + 9015978484853/1828079220031488*s^89 + 1\
```

```
998656668771/609359740010496*s^90 + 1302357194711/609359740010496*s^91 + 4\
985601609901/3656158440062976*s^92 + 345803964649/406239826673664*s^93 + 2\
11032768119/406239826673664*s^94 + 377372506705/1218719480020992*s^95 + 36\
582595315/203119913336832*s^96 + 20740589135/203119913336832*s^97 + 343451\
51005/609359740010496*s^98 + 6144294035/203119913336832*s^99 + 3201590965/\
203119913336832*s^100 + 14552943185/1828079220031488*s^101 + 2370336185/60\
9359740010496*s^102 + 1118165125/609359740010496*s^103 + 1523989775/182807\
9220031488*s^104 + 221642905/609359740010496*s^105 + 92568305/609359740010\
496*s^106 + 110594215/1828079220031488*s^107 + 27869905/1218719480020992*s\
^108 + 9944165/1218719480020992*s^109 + 9979585/3656158440062976*s^110 + 1\
034495/1218719480020992*s^111 + 147935/609359740010496*s^112 + 115105/1828\
079220031488*s^113 + 8855/609359740010496*s^114 + 1771/609359740010496*s^1\
15 + 1771/3656158440062976*s^116 + 77/1218719480020992*s^117 + 7/121871948\
0020992*s^118 + 1/3656158440062976*s^119 + 1
>> [print (20+n,float(f(20+n)), 45+n, float(f(45+n)), 70+n, float(f(70+n)),
   95+n, float(f(95+n))) $ n=0..24]
20, 1.0, 45, 0.999, 70, 0.474, 95, 3.09e-4
21, 1.0, 46, 0.999, 71, 0.423, 96, 1.8e-4
22, 1.0, 47, 0.999, 72, 0.373, 97, 1.02e-4
23, 1.0, 48, 0.998, 73, 0.325, 98, 5.64e-5
24, 1.0, 49, 0.997, 74, 0.279, 99, 3.02e-5
25, 1.0, 50, 0.995, 75, 0.237, 100, 1.58e-5
26, 1.0, 51, 0.993, 76, 0.199, 101, 7.96e-6
27, 1.0, 52, 0.99, 77, 0.164, 102, 3.89e-6
28, 1.0, 53, 0.985, 78, 0.134, 103, 1.84e-6
29, 1.0, 54, 0.979, 79, 0.108, 104, 8.34e-7
30, 1.0, 55, 0.972, 80, 0.0851, 105, 3.64e-7
31, 1.0, 56, 0.962, 81, 0.0663, 106, 1.52e-7
32, 1.0, 57, 0.949, 82, 0.0509, 107, 6.05e-8
33, 1.0, 58, 0.934, 83, 0.0384, 108, 2.29e-8
34, 1.0, 59, 0.915, 84, 0.0285, 109, 8.16e-9
35, 1.0, 60, 0.893, 85, 0.0208, 110, 2.73e-9
36, 1.0, 61, 0.866, 86, 0.0149, 111, 8.49e-10
37, 1.0, 62, 0.836, 87, 0.0105, 112, 2.43e-10
38, 1.0, 63, 0.802, 88, 0.00727, 113, 6.3e-11
39, 1.0, 64, 0.763, 89, 0.00493, 114, 1.45e-11
40, 1.0, 65, 0.721, 90, 0.00328, 115, 2.91e-12
41, 1.0, 66, 0.676, 91, 0.00214, 116, 4.85e-13
42, 1.0, 67, 0.628, 92, 0.00136, 117, 6.32e-14
43, 1.0, 68, 0.577, 93, 8.51e-4, 118, 5.75e-15
44, 0.999, 69, 0.526, 94, 5.2e-4, 119, 2.74e-16
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